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R. No.......... Name Saugat Sigdel .... Class.......... Sec..........

Subject Numerical Methods (CSIT) Date.......... Marks..........

12 pages

Taylor's Series:->

Formal Taylor series for f about c:

$$f(x) \sim f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

$$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \longrightarrow (1)$$

Here, rather than using =, we have written ~ to indicate that we are not allowed to assume that f(x) equals the series on the right. All we have at the moment is a formal series that can be written down provided the successive derivatives $f', f'', f''', \ldots$ exists at the point c.

The series (1) is called "Taylor's series of f at point c".

Taylor's theorem in terms of h:->

If the function f possesses continuous derivatives of order $0, 1, 2, \ldots, (n+1)$ in a closed interval $I = [a, b]$, then for any x in I,

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1} \longrightarrow (1)$$

Where h is any value such that x+h is in I where

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}$$

For some ξ between x and $x+h$.

Notice that h can be positive or negative, the requirement on ξ means $x < \xi < x+h$ if $h > 0$ or $x+h < \xi < x$ if $h < 0$.

The error term E_{n+1} depends on h in two ways: first, h^{n+1} is explicitly present; second, the point ξ generally depends on h. As h converges to zero, E_{n+1} converges to zero with essentially the same rapidity with which h^{n+1} converges to zero. For large n, this is quite rapid. To express this qualitative fact, we can write:

$$E_{n+1} = O(h^{n+1}) \quad \text{as } h \to 0.$$

This is called "big O notation", and it is shorthand for the inequality

$$|E_{n+1}| \leq C|h|^{n+1}$$

where c is a constant. In the present circumstances, this constant could be any number for which $|f^{(n+1)}(t)|/(n+1)! \leq C$, for all t in the initially given interval, I. Roughly $E_{n+1} = O(h^{n+1})$ means that the behavior of E_{n+1} is similar to much simpler expression h^{n+1}.

It is important to realize that equation 1 corresponds to an entire sequence of the theorems, one for each value of n. For example, we can write out the cases n=0,1,2 as follows:

$$f(x+h) = f(x) + f'(\xi_1)h$$
$$= f(x) + O(h)$$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(\xi_2)h^2$$
$$= f(x) + f'(x)h + O(h^2)$$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(x)h^2 + \frac{1}{3!}f'''(\xi_3)h^3$$
$$= f(x) + f'(x)h + \frac{1}{2!}f''(x)h^2 + O(h^3)$$

Errors in Numerical analysis:->

Truncation Errors:->

Truncation errors arise from using an approximation in place of an exact mathematical procedure. Typically it is the error resulting from the truncation of the numerical process. We often use some finite number of terms to estimate the sum of an infinite series. For example,

$S = \sum_{i=0}^{\infty} a_i x^i$ is replaced by the finite sum $\sum_{i=0}^{n} a_i x^i$

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Roundoff Errors:→

Round-off errors occur when a fixed number of digits are used to represent exact numbers. Since the numbers are stored at every stage of computation, round off error is introduced at the end of every arithmetic operation. Consequently, even though an individual round off could be very small, the cumulative effect of a series of computations can be very significant.

Errors in Original data:→

Inherent errors are those that are present in the data supplied to the model. Inherent errors are also known as input errors, contains two components, namely, data errors and conversion errors.

(i) Data errors:→

Data error (also known as empirical error) arises when data for a problem are obtained by some experimental means and are, therefore of limitted accuracy and precision. This may be due to some limitations in instrumentation and reading, and therefore may be unavoidable.

(ii) Conversion Errors:→

Conversion errors (also known as representation errors) arise due to the limitations of the computer to store the data exactly. We know that the floating point representation retains only a specified number of digits. the digits that are not retainde constitute the round-off error.

Notice that the addition e_{x+y} does not mean that error will increase in all cases. It depends on the sign of individual errors. Similar is the case with subtractions.

Since we do not normally know the sign of errors, we can only estimate error bounds. That is we can say that:

$$|e_{x+y}| \leq |e_x| + |e_y|$$

Multiplication:-

Here, we have

$$x_t \times y_t = (x_a + e_x) + (y_a + e_y) = x_a y_a + y_a e_x + x_a e_y + e_x e_y$$

Errors are normally small and their products will be much smaller. Therefore, if we neglect the product of the errors, we get:

$$x_t \times y_t = x_a y_a + x_a e_y + y_a e_x$$

$$= x_a y_a + x_a y_a (e_x/x_a + e_y/y_a)$$

Then, Total error $= e_{xy} = x_a y_a (e_x/x_a + e_y/y_a)$

Division:

We have,

$$\frac{x_t}{y_t} = \frac{x_a + e_x}{y_a + e_y}$$

Multiplying both numerator and denominator by $y_a - e_y$ and rearranging the terms, we get:

$$\frac{x_t}{y_t} = \frac{x_a y_a + y_a e_x - x_a e_y - e_x e_y}{y_a^2 - e_y^2}$$

Dropping all terms that involve only product of errors, we have:

$$\frac{x_t}{y_t} = \frac{x_a y_a + y_a e_x - x_a e_y}{y_a^2} = \frac{x_a}{y_a} + \frac{x_a}{y_a}\left(\frac{e_x}{x_a} - \frac{e_y}{y_a}\right)$$

Thus, Total error $= e_{x/y} = \frac{x_a}{y_a}\left(\frac{e_x}{x_a} - \frac{e_y}{y_a}\right)$

Also, for ~~tol~~ Multiplication and division:

$$e_{x/y} \leq \left|\frac{x_a}{y_a}\right|\left(\left|\frac{e_x}{x_a}\right| + \left|\frac{e_y}{y_a}\right|\right)$$

$$e_{xy} \leq |x_a y_a|\left(\left|\frac{e_x}{x_a}\right| + \left|\frac{e_y}{y_a}\right|\right)$$

Relating Newton's Method to other Methods:->

- The formula for Method of false position (interpolation) Method is given by:

$$x_2 = x_1 - f(x_1)\frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$$

- Which can also be written as:

$$x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{(x_n - x_{n-1})}}$$

- We see that the denominator of the factorial term is exactly the definition of derivative except not taken to the limit as the two x-values approaches each other.
- Because the denominator of the fraction term is an approximation to the derivative of f, we see the close resemblance to Newton Method.
- Similarly for secant method, it has exactly this same resemblance to Newton's Method because it is just linear Interpolation without the requirement that two x-values bracket the root.

Newton's method for polynomials:->

Synthetic division:->

Suppose we want to find the value at $x = 2$ of $P(x) = 2x^3 + x^2 - 3x - 3$, write the coefficients in row and follow this pattern:

$x=2$	2	1	-3	-3
		4	10	14
	2	5	7	11

- The last row of numbers is the coefficients of the reduced polynomial and the remainder from the division.
- The final result 11 is the value of the polynomial at $x=2$. This is also remainder form for the division:

$$\frac{2x^3+x^2-3x-3}{(x-2)} = 2x^2+5x+7+\frac{11}{x-2}$$

Horner's Method and synthetic division are precisely the same. The value of synthetic division in getting a root by Newton's method is that, if the reduced polynomial is divided by $(x-2)$, the remainder from this is the value of the derivative at $x=2$.

$x=2$	2	5	7
		4	10
	2	9	25

where circled 25 is $p'(2)$.

- With the values of $p(2)$ and $p'(2)$ available, we can use them in Newton's method to estimate root starting with $x_1=2$.

Synthetic division algorithm & remainder theorem:-

- We can develop an algorithm for synthetic division and show that the remainders are the value of the polynomial by writing nth degree polynomial as:

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1 x + a_0$$

We wish to divide by the factor $(x-x_1)$ giving a reduced polynomial $Q_{n-1}(x)$ of degree $(n-1)$, and a remainder R, which is constant:

$$\frac{P_n(x)}{x-x_1} = Q_{n-1}(x) + \frac{R}{(x-x_1)}$$

Rearranging:

$$P_n(x) = (x-x_1) Q_{n-1}(x) + R$$

At $x = x_1$

$$P_n(x_1) = 0\left[Q_{n-1}(x_1)\right] + R = R$$

which is the remainder theorem. The remainder on division by $(x-x_1)$ is the value of the polynomial at $x = x_1$, $P_n(x_1)$.

If we differentiate $P_n(x)$, we get:

$$P_n'(x) = (x-x_1) Q_{n-1}'(x) + (1) Q_{n-1}(x) + 0$$

Letting $x = x_1$, we have:

$$P_n'(x_1) = 0\left[Q_{n-1}'(x)\right] + Q_{n-1}(x)$$

$$\therefore P_n'(x_1) = Q_{n-1}(x)$$

We evaluate the Q polynomial at x_1 by a second division whose remainder equals $Q_{n-1}(x_1)$. This verify that the second remainder from synthetic division yields yields the value for derivative of the polynomial.

We now develop the synthetic division algorithm, writing $Q_{n-1}(x)$ in the form similar to $P_n(x)$:

$$P_n(x) = (x-x_1) Q_{n-1}(x) + R = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

$$= (x-x_1)\left[b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \cdots + b_1 x + b_0\right] + R$$

Napping out and equating the coefficients of like terms in x, we get:

coefficient of x^n : $a_n = b_{n-1}$

x^{n-1} : $a_{n-1} = b_{n-2} - x_1 b_{n-1}$

$\vdots$

x : $a_1 = b_0 - x_1 b_1$

constant : $a_0 = R - x_1 b_0$

or

$b_{n-1} = a_n$

$b_{n-2} = a_{n-1} + x_1 b_{n-1}$

$b_{n-3} = a_{n-2} + x_1 b_{n-2}$

$\vdots$

$b_0 = a_1 - x_1 b_1$

$R = a_0 + x_1 b_0$

The general form is $b_i = a_{i+1} + x_1 b_{i+1}$, by which all b's may be calculated, provided we first set $b_n = 0$. If this is compared to the preceeding synthetic divisions, it is seen to be identical, except that we now have a vertical array.

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Solution of ALGEBRAIC AND TRANSCEDENTAL EQUATIONS:

Algebraic Equation:→

- An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ where a's are constants ($a_0 \neq 0$) and n is a positive integer, is called a polynomial in x of degree n.
- The polynomial $f(x) = 0$ is called an algebraic equation of degree n.
- If f(x) contains some other functions such as trigonometric, logarithmic, Exponential etc, then $f(x) = 0$ is called transcedental equation.

Root:→

- The value α of x which satisfies $f(x) = 0$ is called root of $f(x) = 0$. Geometrically, a root of $f(x) = 0$ is that value of x where the graph of $y = f(x)$ crosses the x-axis.
- The process of finding root of an equation is known as the solution of that equation.

Iterative Methods of Solution of equation:→

- An iterative Methods begins with an approximate value of the root. This initial approximation is then successively improved iteration by iteration and this process is stopped when the desired level of accuracy is achieved.
- The various iterative Methods begin their process with one or More initial approximations. Based on the number of initial approximations used, these iterative Methods are divided into two categories:
  (i) Bracketing Methods
  (ii) Open-end Methods

(i) Bracketing Methods:->

- Bracketing Methods begin with two initial approximations which bracket the root. Then the width of this bracket is symmetrically reduced until the root is reached to desired accuracy.
- The commonly used methods in this category are:

(1) Graphical Method

(2) Bisection Method

(3) Method of false position

(ii) Open end Methods:->

- Open-end methods are used on formulae which require a single starting value or two starting values which do not necessarily bracket the root.
- The following methods fall under this category:

(1) Secant Method

(2) Iteration Method

(3) Newton Raphson Method

(4) Muller's Method

(5) Horner's Method

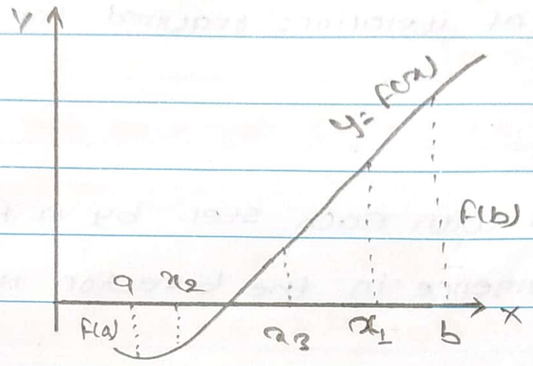
(6) Lin-Bairstow Method.

Intermediate value property:->

If $f(x)$ is continuous in the interval $[a,b]$ and $f(a)$ and $f(b)$ have different signs, then the equation $f(x)=0$ has atleast one root between $x=a$ and $x=b$.

1. Bisection Method :→

- This Method is based on the repeated application of the inter-Mediate value property.



- Let the function $f(x)$ be Continuous between a and b. for definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the first approximation to the root is:

$$x_1 = \frac{1}{2}(a+b)$$

- If $f(x_1) = 0$, then x_1 is a root of $f(x)=0$. Otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.
- In above fig, $f(x_1)$ is +ve, so that the root lies between a and x_1. Then the second approximation to the root is $x_2 = \frac{1}{2}(a+x_1)$. If $f(x_2)$ is -ve, the root lies between x_1 and x_2. Then third approximation to the root is:

$$x_3 = \frac{1}{2}(x_1+x_2), \text{ And so on...}$$

Observation :→

- Since, the new interval containing the root, is exactly half the length of the previous one, the interval width is reduced by a factor of $\frac{1}{2}$ at each step. At the end of the nth step, the new interval will therefore, be of length $(b-a)/2^n$. If on repeating this process n times, the latest interval is as small as given ϵ, then $(b-a)/2^n \leq \epsilon$

or, $n \geq [\log(b-a) - \log\epsilon]/\log 2$

- This gives the number of iterations required for achieving an accuracy ϵ.

Rate of convergence :->

- As the error decreases with each step by a factor of $\frac{1}{2}$, (i.e. $e_{n+1}/e_n = \frac{1}{2}$), the convergence in the bisection method is linear.

Q. Find a root of a equation $x^3-4x-9=0$, using bisection method correct to three decimal places.

Solution :->

$f(x) = x^3-4x-9 = 0$

$f(3) = 6$, $f(2) = -9$. Therefore the root lies between 2 & 3.

Now, Tabulating the datas formed by the process of bisection method:

S.N	a	b	$x = \frac{a+b}{2}$	f(a)	f(b)	f(x)	Remarks
1.	2	3	2.5	-9	6	-3.375	$a = x$ ($\because f(x)<0$)
2.	2.5	3	2.75	-3.375	6	0.796	$b = x$ ($\because f(x)>0$)
3.	2.5	2.75	2.625	-3.375	0.796	-1.41	$a = x$ ($\because f(x)<0$)
4.	2.625	2.75	2.6875	-1.412	0.796	-0.339	$a = x$ ($\because f(x)<0$)
5.	2.6875	2.75	2.7188	-0.3391	0.7969	0.2218	$b = x$ ($\because f(x)>0$)
6.	2.6875	2.7188	2.7032	-0.3391	0.2218	-0.0597	$a = x$ ($\because f(x)<0$)
7.	2.7032	2.7188	2.7110	-0.0597	0.2218	0.0806	$b = x$ ($\because f(x)>0$)
8.	2.7032	2.7110	2.7071	-0.0597	0.0806	0.0103 ~~0.0806~~	$b = x$ ($\because f(x)>0$)
9.	2.7032	2.7071	2.7052	-0.0597	0.0103	-0.0239	$a = x$ ($\because f(x)<0$)
10.	2.7052	2.7071	2.7062	-0.0239	0.0103	-0.0059	$a = x$ ($\because f(x)<0$)
11.	2.7062	2.7071	2.7067	-0.0059	0.0103	0.0031	$b = x$ ($\because f(x)>0$)
12.	2.7062	2.7067	2.7065	-0.0059	0.0031	-0.0005	Required Root

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Q.. Using Bisection Method, find an approximate root of the equation $\sin x = 1/x$, that lies between $x=1$ and $x=1.5$ (Measured in radians). Carry out computations upto 7^{th} stage.

[Ans = 1.11328]

Q.. Find the root of the equation $\cos x = xe^x$ using bisection method correct to four decimal places.

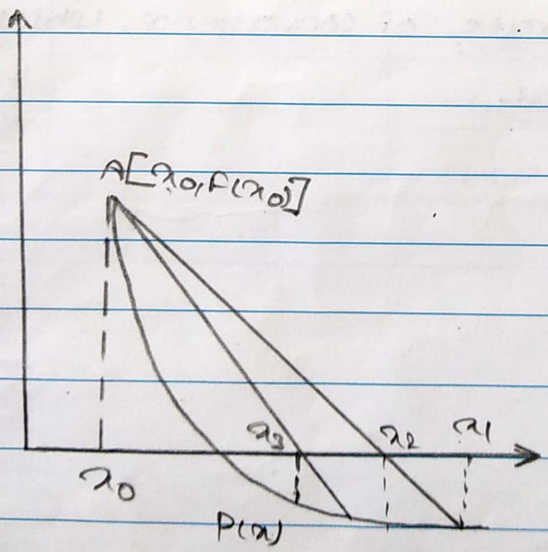
[Ans = 0.5156]

Q.. Find a positive real root of $x\log_{10}x = 1.2$ using the bisection method. Correct to 3 decimal places.

[Ans = 2.687]

Method of false position or Regula falsi method or interpolation method:→

- Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. This indicates that the root lies between x_0 and x_1 and consequently $f(x_0)f(x_1)<0$.



Equation of the chord joining points $A[x_0, f(x_0)]$, $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x - x_0)$$

- The method consists in replacing the curve AB by means of the chord AB and taking the ~~intersection~~ point of intersection of the chord with the x-axis as an approximation to the root. So, the abscissa of the point where the chord cuts the x-axis $(y=0)$ is given by :

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \longrightarrow (1)$$

which is the approximation to the root.

- If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in (1), we obtain the next approximation x_3. (The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly).
- This procedure is repeated till the root is found to the desired accuracy. The iteration process based on (1) is known as the method of false position.

Rate of convergence :-

This method has a linear rate of convergence, which is faster than that of bisection method.

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Q. Find a real root of the equation $x^3-2x-5=0$ by the method of false position correct to three decimal places.

Solution:

We have $f(x)=x^3-2x-5=0$

$f(2)=-1$ & $f(3)=16$. Hence root lies between -1 & 3.

Now tabulating the datas formed by the process of method of false position, we get:

S.N	a	b	$x=a-\frac{(b-a)f(a)}{f(b)-f(a)}$	f(a)	f(b)	f(x)	Remarks
1	2	3	2.0588	-1	16	-0.3908	$a=x$ ($\because f(x)<0$)
2.	2.0588	3	2.0812	-0.3908	16	-0.1475	$a=x$ ($\because f(x)<0$)
3.	2.0812	3	2.0896	-0.1475	16	-0.0552	$a=x$ ($\because f(x)<0$)
4.	2.0896	3	2.0927	-0.0552	16	-0.0203	$a=x$ ($\because f(x)<0$)
5.	2.0927	3	2.0938	-0.0203	16	-0.0078	$a=x$ ($\because f(x)<0$)
6.	2.0938	3	2.0942	-0.0078	16	-0.0035	$a=x$ ($\because f(x)<0$)
7.	2.0942	3	2.0944	-0.0035	16	-0.0017	$a=x$ ($\because f(x)<0$)
8.	2.0944	3	2.0945	-0.0017	16	0.0006	Root

Q. Find the root of the equation $\cos x = xe^x$ using Regula-falsi method correct to four decimal places.

[Ans = 0.5177]

Q. Find real root of equation $x\log_{10}x = 1.2$ by Regula-falsi method correct to four decimal places.

[Ans = 2.7406]

Q. Use the method of false position to find fourth root of 32 correct to three decimal places.

[Ans = 2.378]

Secant Method:→

This method is an improvement over the method of false position as it doesnot require the condition of $f(x_0) f(x_1) < 0$ of that method. Here also the graph of the function $y = f(x)$ is approximated by a secant line but at each iteration, two most recent approximations to the root are used to find the next approximation. Also, it is not necessary that the interval must contain the root.

Taking x_0, x_1 as the initial limits of the ~~approximation~~ interval, we write the equation of the chord joining these as:

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_1)$$

Then the abscissa of the point where it crosses the x-axis $(y=0)$ is given by:

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

which is an approximation to the root. The general formula for successive approximations is, therefore, given by:

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

Drawbacks:→

If at any iteration $f(x_n) = f(x_{n-1})$, this method fails and shows that it does not converge necessarily. This is a drawback of secant method over the method of false position which always converges.

Assignment:→ 1

Prove that the convergence of secant method is 1.6.

Q. Find a root of the equation $x^3-2x-5=0$ using secant Method. Correct to three decimal places.

Solution:

$f(x) = x^3-2x-5$

$f(2) = -1$, $f(3) = 16$

Tabulating the datas formed by secant Method, we get:

S.N	x_0	x_1	$x_2 = x_1 - \frac{x_1-x_0}{f(x_1)-f(x_0)} f(x_1)$	Remarks	$f(x_2)$
1	2	3	2.0588	$x_1 = x_2$ $x_0 = x_1$	-0.3907
2.	3	2.0588	2.0812	$x_1 = x_2$ $x_0 = x_1$	-0.1475
3.	2.0588	2.0812	2.0947	$x_1 = x_2$ $x_0 = x_1$	0.0017
4.	2.0812	2.0947	2.0945	Required root	

Q. Find the root of equation $xe^x = \cos x$ using the Secant Method. Correct to 4 decimal places.

[Ans: 0.5177]

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Newton-Raphson Method

Let x_0 be an approximate root of the equation $f(x)=0$. If $x_1 = x_0+h$ be the exact root, then $f(x_1)=0$.

$\therefore$ Expanding $f(x_0+h)$ by Taylor's series $f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots=0$

Since h is small, neglecting h^2 and higher powers of h, we get

$f(x_0)+hf'(x_0)=0$

$$\therefore h = -\frac{f(x_0)}{f'(x_0)} \longrightarrow (1)$$

$\therefore$ A closer approximation to the root is given by:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly starting with x_1, a still better approximation x_2 is given by:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n=0,1,2,\cdots) \longrightarrow (2)$$

Observations:→

Observation 1:⇒

Newton's Method is useful in cases of large values of $f'(x)$ i.e. ~~while~~ when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

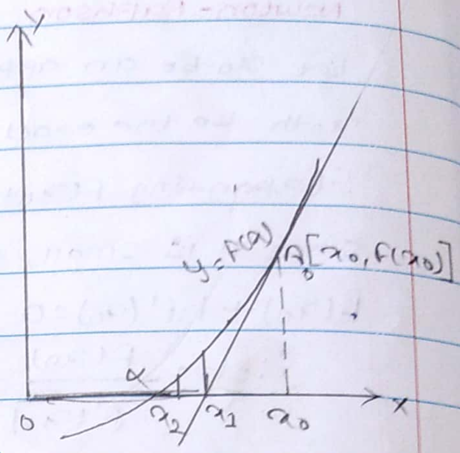
For if $f'(x)$ is small in the vicinity of the root, then by (1), h will be large and the computation of the root is slow or may not be possible. Thus this method is not suitable in those cases where the graph of $f(x)$ is nearly horizontal while crossing the x-axis.

Observation 2 :→ Geometrical interpretation :

Let x_0 be the point near the root α of the equation $f(x)=0$. Then the equation of the tangent at $A_0[x_0, f(x_0)]$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

It cuts x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$



which is the first approximation to the root α.

If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis at x_2 which is nearer to α and is, therefore a second approximation to the root. Hence the method consists in replacing the part of the curve between the point A_0 and the x-axis by means of the tangent to the curve at A_0.

Convergence of Newton-Raphson Method :→

Newton's formula converges provided the initial approximation x_0 is chosen sufficiently close to the root.

If it is not near the root, the procedure may lead to an endless cycle. A bad initial choice will lead to one astray. Thus a proper choice of initial guess is very important for the success of the Newton's method.

we have :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2, \cdots) \longrightarrow (1)$$

Comparing (1) with the relation $x_{n+1} = \phi(x_n)$ of the iteration method, we get :

$$\phi(x_n) = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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In general, $\phi(x) = x - \frac{f(x)}{f'(x)}$ which gives $\phi'(x) = \frac{f(x) f''(x)}{[f'(x)]^2}$

Since the iteration Method Converges if $|\phi'(x)| < 1$

$\therefore$ Newton's formula will converge if $|f(x) f''(x)| < |f'(x)|^2$ in the interval considered.

Assuming $f(x)$, $f'(x)$ and $f''(x)$ to be continuous, we can select small interval in the vicinity of the root α, in which above condition is satisfied.

Newton's method converges conditionally while Regula-falsi Method always converges. However when once Newton-Rapheon Method converges, it converges faster and is preferred.

Newton's Method has a quadratic convergence :-

Suppose x_n differs from the root α by a small quantity ϵ_n so that

$$x_n = \alpha + \epsilon_n \text{ and } x_{n+1} = \alpha + \epsilon_{n+1}$$

Then the equation $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ becomes

$$\alpha + \epsilon_{n+1} = \alpha + \epsilon_n - \frac{f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)}$$

$$\epsilon_{n+1} = \epsilon_n - \frac{f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)}$$

$$= \epsilon_n - \frac{f(\alpha) + \epsilon_n f'(\alpha) + \frac{1}{2!}\epsilon_n^2 f''(\alpha) + \cdots}{f'(\alpha) + \epsilon_n f''(\alpha) + \cdots} \quad \text{By Taylor's Expansion}$$

$$= \epsilon_n - \frac{\epsilon_n f'(\alpha) + \frac{1}{2}\epsilon_n^2 f''(\alpha) + \cdots}{f'(\alpha) + \epsilon_n f''(\alpha) + \cdots} = \frac{\epsilon_n^2 f''(\alpha)}{2 f'(\alpha)} \quad [\because f(\alpha) = 0]$$

$$= \frac{\epsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)}$$

This shows that the subsequent error at each step, is proportional to the square of the previous error and as such the convergence is quadratic. Thus Newton-Raphson Method has second order convergence.

Q. Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson Method.

Solution:

$f(x) = x^4 - x - 10$ $\quad f'(x) = 4x^3 - 1$

$f(1) = -10 = -ve$, $f(2) = 4 = +ve$

$\therefore$ Root lies between 1 & 2.

S.N	x_n	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_n)$	$f'(x_n)$	$f(x_{n+1})$	$f'(x_{n+1})$
1.	2	1.5806	13	31		
2.	1.5806	1.3331	3.6616	14.7966		
3.	1.3331	1.2357	0.8252	8.4765		
4.	1.2357	1.2210	0.0962	6.5483		
5.	1.2210	1.2207	0.0017	6.2814		
6.	1.2207	1.2207	-0.0001	6.2764		

Q. Find by Newton's Method, the real root of the equation $3x = \cos x + 1$ correct to four decimal places. [Ans = 0.6071]

Q. Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places. [Ans: 2.74065]

Q. Using Newton's Iteration formula ~~find~~ derive the following:

(i) Iterative formula to find $\frac{1}{N}$ is $x_{n+1} = x_n(2 - Nx_n)$

(ii) Iterative formula to find $\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$

(iii) Iterative formula to find $1/\sqrt{N}$ is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$

(iv) Iterative formula to find $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + N/x_n^{k-1}\right]$

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(i) Iterative Formula to find $1/N$ is $x_{n+1} = x_n(2 - Nx_n)$

proof:

Let $x = \frac{1}{N}$ or, $\frac{1}{x} - N = 0$

Taking $f(x) = 1/x - N$, we have $f'(x) = -x^{-2}$

Then Newton's Formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n^{-2}} = x_n + \left(\frac{1}{x_n} - N\right)x_n^2$$

$$= x_n + (x_n - Nx_n^2)$$

$$= 2x_n - Nx_n^2 = x_n(2 - Nx_n)$$

(ii) Iterative Formula to find $\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$

Proof:

Let $x = \sqrt{N}$ or, $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Then Newton's formula gives:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}(x_n + N/x_n)$$

(iii) Iterative formula to find $1/\sqrt{N}$ is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$

Let $x = \frac{1}{\sqrt{N}}$ or, $x^2 - \frac{1}{N} = 0$

Taking $f(x) = x^2 - \frac{1}{N}$, we have $f'(x) = 2x$

Then Newton's Formula gives:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 1/N}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

~~(iv) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$~~

(iv) Iterative formula to find $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + N/x_n^{k-1}\right]$

proof:

Let $x = \sqrt[k]{N}$ or, $x^k - N = 0$

Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$

Then Newton's formula gives:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \frac{x_n^k - N}{k x_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]$$

Q. Evaluate the following (correct to four decimal places) by Newton's iteration method.

(i) $1/31$ (ii) $\sqrt{5}$ (iii) $1/\sqrt{14}$ (iv) $\sqrt[3]{24}$ (v) $(30)^{-1/5}$

Solution:

(i) Taking $N=31$ for formula to find $1/N$. i.e. $x_{n+1} = x_n(2-Nx_n)$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$

Then, $x_1 = x_0(2-31x_0) = 0.03(2-31\times0.03) = 0.0321$

$x_2 = x_1(2-31x_1) = 0.0321(2-31\times0.0321) = 0.032257$

$x_3 = x_2(2-31x_2) = 0.032257(2-31\times0.032257) = 0.03226$

Since $x_2 = x_3$ upto 4 decimal places, we have $1/31 = 0.0323$

(ii) $\sqrt{5}$, taking $N=5$ for formula to find $\sqrt{N}$. i.e. $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$

Since an approximate value of $\sqrt{5} = 2$, we take $x_0 = 2$.

Then, $x_1 = \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right) = \frac{1}{2}\left(2+\frac{5}{2}\right) = 2.25$

$x_2 = \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right) = \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) = 2.2361$

$x_3 = \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right) = \frac{1}{2}\left(2.2361 + \frac{5}{2.2361}\right) = 2.2361$

Since, $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{5} = 2.2361$

(iii) Taking $N=14$ for formula to find $1/\sqrt{14}$. i.e. $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$

Since an approximate value of $\frac{1}{\sqrt{14}} = \frac{1}{\sqrt{16}} = \frac{1}{4} = 0.25$, we take $x_0 = 0.25$.

Then $x_1 = \frac{1}{2}\left[x_0 + (14x_0)^{-1}\right] = \frac{1}{2}\left[0.25 + (14\times0.25)^{-1}\right] = 0.26785$

$x_2 = \frac{1}{2}\left[x_1 + (14x_1)^{-1}\right] = \frac{1}{2}\left[0.26785 + (14\times0.26875)^{-1}\right] = 0.2672618$

$x_3 = \frac{1}{2}\left[x_2 + (14x_2)^{-1}\right] = \frac{1}{2}\left[0.2672618 + (14\times0.2672618)^{-1}\right] = 0.2672612$

Since $x_2 = x_3$ upto 4 decimal places, we take $\frac{1}{\sqrt{14}} = 0.2673$

for $\sqrt[k]{N}$ $\quad x_{n+1} = \frac{1}{k}\left[(k-1)x_n + N/x_n^{k-1}\right]$

(iv) Taking $N=24$ and $k=3$, the above formula [redacted] becomes $x_{n+1} = \frac{1}{3}\left[2x_n + 24/x_n^2\right]$

Since an approximate value of $(24)^{1/3} \approx (27)^{1/3} = 3$, we take $x_0 = 3$.

Then $x_1 = \frac{1}{3}(2x_0 + 24/x_0^2) = \frac{1}{3}(6 + 24/9) = 2.88889$

$x_2 = \frac{1}{3}(2x_1 + 24/x_1^2) = \frac{1}{3}\left[2\times 2.88889 + 24/(2.88889)^2\right] =$ [redacted] 2.88451

$x_3 = \frac{1}{3}(2x_2 + 24/x_2^2) = \frac{1}{3}\left[2\times 2.88451 + 24/(2.88451)^2\right] = 2.8845$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{1/3} = 2.8845$

(v) Taking $N=30$ and $k=-5$, the above formula for $\sqrt[k]{N}$ ~~beco~~ i.e.

$x_{n+1} = \frac{1}{k}\left[(k-1)x_n + N/x_n^{k-1}\right]$ becomes $x_{n+1} = -\frac{1}{5}\left(6x_n + 30/x_n^{-6}\right) = \frac{x_n}{5}(6 - 30x_n^5)$

Since an approximate value of $(30)^{-1/5} = (32)^{-1/5} = \frac{1}{2} = 0.5$, we take $x_0 = \frac{1}{2}$

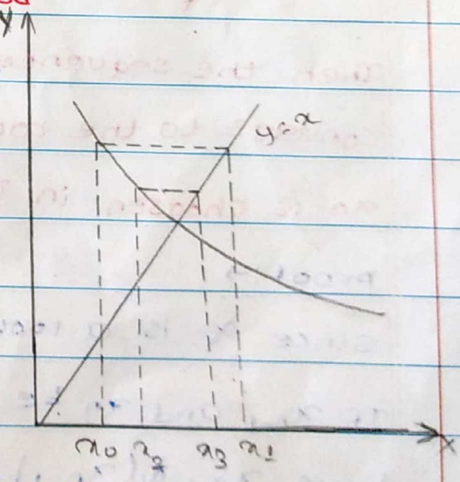
Then, $x_1 = \frac{x_0}{5}(6 - 30x_0^5) = \frac{1}{10}(6 - 30/2^5) = 0.50625$

$x_2 = \frac{x_1}{5}(6 - 30x_1^5) = \frac{1}{10}(6 - 30(0.50625)^5) = 0.506495$

$x_3 = \frac{x_2}{5}(6 - 30x_2^5) = \frac{1}{10}(6 - 30(0.506495)^5) = 0.506496$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{-1/5} = 0.5065$

Fixed Point Iteration Method / Iteration Method.



To find roots of the equation $f(x)=0$ (i) by successive approximations, we rewrite (i) in the form of $x = \phi(x)$.

The roots of (i) are the same as the points of intersection of the straight line $y=x$ and the curve representing $y=\phi(x)$. Figure represents the working of the ~~spiral~~ iteration method which provides a spiral solution.

Let $x = x_0$ be initial approximation of the ~~initial~~ desired root α. Then the first approximation x_1 is given by:

$$x_1 = \phi(x_0)$$

Now, treating x_1 as the initial value, the second approximation is:

$$x_2 = \phi(x_1)$$

Proceeding this way, the n^{th} approximation is given by:

$$x_n = \phi(x_{n-1})$$

Sufficient Condition for Convergence of iterations:-

Now, it is not sure whether the sequence of approximations $x_1, x_2, \dots x_n$ always converges to the same number which is the root of equation or not. As such, we have to choose the initial approximation x_0 suitably so that the successive approximations $x_1, x_2, \dots x_n$ converges to root α. The following theorem helps in making right choice of x_0.

Theorem:-

If (i) α be a root of $f(x)=0$ which is equivalent to $x = \phi(x)$

(ii) I be any interval containing the point $x = \alpha$.

(iii) $|\phi'(x)| < 1$ for all x in I,

Then the sequence of approximations $x_0, x_1, x_2, \dots x_n$ will converge to the root α provided the initial approximation x_0 is choosen in I.

Proof:-

Since α is a root of $x = \phi(x)$, we have $\alpha = \phi(\alpha)$

If x_{n-1} and x_n be two successive approximations to α, we have $x_n = \phi(x_{n-1})$

$$\therefore x_n - \alpha = \phi(x_{n-1}) - \phi(\alpha) \longrightarrow (i)$$

By mean value theorem, $\dfrac{\phi(x_{n-1}) - \phi(\alpha)}{x_{n-1} - \alpha} = \phi'(\xi)$ where $x_{n-1} < \xi < \alpha$

Hence (i) becomes:

$$x_n - \alpha = (x_{n-1} - \alpha)\, \phi'(\xi)$$

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If $|\phi'(x_i)| \le k < 1$ for all i, then

$$|x_n-\alpha| \le k|x_{n-1}-\alpha|$$

$$|x_{n-1}-\alpha| \le k|x_{n-2}-\alpha|$$

$$|x_n-\alpha| \le k^2|x_{n-2}-\alpha|$$

i.e., preceeding this way, $|x_n-\alpha| \le k^n|x_0-\alpha|$

As $n\to\infty$, the RHS tends to zero, therefore, the sequence of approximations converges to root α.

Q. Find a real root of the equation $\cos x = 3x-1$ correct to three decimal places using:

(i) Iteration Method.

Solution:

We have $f(x) = \cos x - 3x + 1 = 0$

$f(0) = 2 = +ve$ and $f(\pi/2) = -3\pi/2 + 1 = -ve$

$\therefore$ A root lies between 0 and $\pi/2$

Rewriting the given equation as $x = \frac{1}{3}(\cos x + 1) = \phi(x)$, we have

$$\phi'(x) = \frac{\sin x}{3} \text{ and } |\phi'(x)| = \frac{1}{3}|\sin x| < 1 \text{ in } (0, \pi/2).$$

Hence the iteration method can be applied and we start with $x_0 = 0$. Then successive approximations are,

$$x_1 = \phi(x_0) = \frac{1}{3}(\cos 0 + 1) = 0.6667$$

$$x_2 = \phi(x_1) = \frac{1}{3}(\cos 0.6667 + 1) = 0.5953$$

$$x_3 = \phi(x_2) = \frac{1}{3}(\cos 0.5953 + 1) = 0.6093$$

$$x_4 = \phi(x_3) = \frac{1}{3}(\cos 0.6093 + 1) = 0.6067$$

$$x_5 = \phi(x_4) = \frac{1}{3}(\cos 0.6067 + 1) = 0.6072$$

$$x_6 = \phi(x_5) = \frac{1}{3}(\cos 0.6072 + 1) = 0.6071$$

Hence x_5 and x_6 being almost the same, the root is 0.607 correct to 3 decimal places.

Q. Using iteration Method, find a root of the equation $x^3+x^2-1=0$ correct to four decimal places.

Solution:

we have $f(x)=x^3+x^2-1=0$

since $f(0)=-1$ and $f(1)=1$, a root lies between 0 and 1.

Rewriting the given equation as $x=(x+1)^{-1/2}=\phi(x)$, we have $\phi'(x)=-\frac{1}{2}(x+1)^{-3/2}$ and $|\phi'(x)|<1$ for $x<1$. Hence the iteration Method can be applied. Starting with $x_0=0.75$, the successive approximations are:

$$x_1=\phi(x_0)=\frac{1}{\sqrt{x_0+1}}=0.7559$$

$$x_2=\phi(x_1)=\frac{1}{\sqrt{0.7559+1}}=0.75466$$

$$x_3=0.75492,\ x_4=0.75487,\ x_5=0.75488$$

Hence, x_4 and x_5 being almost the same, the root is 0.7548 correct to 4 decimal places.

Q. Using iteration Method, find a root of the equation $x^3+x^2-1=0$ correct to four decimal places.

Solution:

we have $f(x)=x^3+x^2-1=0$

Since $f(0)=-1$ and $f(1)=1$, a root lies between 0 and 1

Rewriting the given equation as $x=(x+1)^{-1/2}=\phi(x)$, we have $\phi'(x)=-\frac{1}{2}(x+1)^{-3/2}$ and $|\phi'(x)|<1$ for $x<1$. Hence the Iteration Method can be applied. Starting with $x_0=0.75$, the successive approximations are:

$$x_1=\phi(x_0)=\frac{1}{\sqrt{x_0+1}}=0.7559$$

$$x_2=\phi(x_1)=\frac{1}{\sqrt{0.7559+1}}=0.75466$$

$$x_3=0.75492,\ x_4=0.75487,\ x_5=0.75488$$

Hence, x_4 and x_5 being almost the same, the root is 0.9548 correct to 4 decimal places.

Q. Apply iteration method to find the negative root of the equation $x^3-2x+5=0$ correct to four decimal places.

Solution:

If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the root of:

$$(-x)^3-2(-x)+5=0$$

∴ Negative root of the given equation is the positive root of

$$f(x)=x^3-2x-5=0$$

Since $f(2)=-1$ and $f(3)=16$, a root lies between 2 and 3.

Rewriting (i) as $x=(2x+5)^{1/3}=\phi(x)$, we have:

$$\phi'(x)=\frac{1}{3}(2x+5)^{-2/3} \text{ and } |\phi'(x)|<1 \text{ for } x<3.$$

∴ The iteration method can be applied.

With $x_0=2$, the successive approximations are:

$$x_1=\phi(x_0)=(2x_0+5)^{1/3}=2.08008$$

$$x_2=\phi(x_1)=2.09235$$

$$x_3=2.09422,\ x_4=2.09450,\ x_5=2.09454$$

Since x_4 and x_5 being almost the same, the root of (1) is 2.0945 correct to 4 decimal places.

Hence the negative root of the given equation is -2.0945

Q. Find a real root of $2x-\log_{10}x=7$ correct to four decimal places using iteration method.

Solution: We have $f(x)=2x-\log_{10}x-7$

$$f(3)=-1.4471\ ;\ f(4)=0.398$$

∴ A root lies between 3 and 4

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Rewriting the given equations as $x = \frac{1}{2}(\log_{10}x + 7) = \phi(x)$, we have

$$\phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10}e\right) \qquad [\because \log_{10}e = 0.4343]$$

$\therefore |\phi'(x)| < 1$ when $3 < x < 4$

since $|f(4)| < |f(3)|$, the root is near to 4.

Hence, the iteration method can be applied taking $x_0 = 3.6$, the successive approximations are:

$$x_1 = \phi(x_0) = \frac{1}{2}(\log_{10}3.6 + 7) = 3.77815$$

$$x_2 = \phi(x_1) = \frac{1}{2}(\log_{10}3.77815 + 7) = 3.78863$$

$$x_3 = \phi(x_2) = \frac{1}{2}(\log_{10}3.78863 + 7) = 3.78924$$

$$x_4 = \phi(x_3) = \frac{1}{2}(\log_{10}3.78924 + 7) = 3.78927$$

Hence x_3 and x_4 being almost equal, the root is 3.7892 correct to 4 decimal places.

Q. Find the smallest root of the equation:

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \cdots = 0$$

Solution: writing the given equation as:

$$x = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \cdots = \phi(x)$$

Omitting x^2 and higher powers of x, we get $x = 1$ approximately

Taking $x_0 = 1$, we obtain:

$$x_1 = \phi(x_0) = 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \frac{1}{(5!)^2} + \cdots = 1.2239$$

$$x_2 = \phi(x_1) = 1 + \frac{(1.2239)^2}{(2!)^2} - \frac{(1.2239)^3}{(3!)^2} + \frac{(1.2239)^4}{(4!)^2} - \frac{(1.2239)^5}{(5!)^2} + \cdots = 1.3263$$

Similarly:

$x_3 = 1.38 \quad x_4 = 1.409 \quad x_5 = 1.425$

$x_6 = 1.434 \quad x_7 = 1.439 \quad x_8 = 1.442$

The values of x_7 and x_8 indicate that the root is 1.44 correct to 2 decimal places.